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SMOOTH STRUCTURES ON SOME OPEN 4-MANIFOLDS

FAN DING[†]

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Let M be the total space of a smooth oriented $\mathbf{R}^2$-bundle over a connected closed oriented surface S. Then there exists an uncountable family of diffeomorphism classes of oriented 4-manifolds which are homeomorphic to M.

1. INTRODUCTION

In 1982, Michael Freedman startled the topological community by pointing out the existence of an "exotic $\mathbf{R}^4$", a smooth manifold homeomorphic to $\mathbf{R}^4$, but not diffeomorphic to it. Gompf [1] has shown that there are four smooth structures on $\mathbf{R}^4$ by using Michael Freedman's work [2] and Simon Donaldson's theorem [3]. Subsequently, in [4], Gompf has shown that there exists an uncountable, doubly indexed family of exotic $\mathbf{R}^4$'s by using Taubes' result. Recently, in [5], Gompf has proven that for any topological 4-manifold M, M − pt. admits uncountably many smoothings, and he has conjectured that any noncompact 4-manifold can be smoothed in uncountably many distinct ways.

Let M be the total space of a smooth oriented $\mathbf{R}^2$-bundle over a closed connected oriented surface S. In Section 2, we apply the method of Gompf [4] to obtain the following result.

THEOREM 1. *There exists an uncountable family of diffeomorphism classes of oriented 4-manifolds which are homeomorphic to M.*

After proving Theorem 1, we give some remarks. The result stated in Remark 3 generalizes Theorem 1.

It is shown in Section 3 that:

THEOREM 2. *Let M be a compact simply-connected topological 4-manifold, its boundary $\partial M \neq \emptyset$. Then there exists a complex structure on the interior of M.*

From Theorem 2, an exotic $\mathbf{CP}^2 \# \mathbf{CP}^2$ − pt., a smooth manifold homeomorphic to $\mathbf{CP}^2 \# \mathbf{CP}^2$ − pt., but not diffeomorphic to it, is constructed.

We use the following terminology and notation.

$\mathbf{R}^4$ will denote Euclidean 4-space with the standard smooth structure. We will unless otherwise noted, work in the category of oriented smooth manifolds and orientation-preserving diffeomorphisms. All codimension zero embeddings will be assumed to preserve orientation. If M is an oriented manifold, $\overline{M}$ will denote the manifold obtained from M by reversing orientation.

[†]Current address: Department of Mathematics, Peking University, Beijing 100871, Peoples' Republic of China.

Definition. Let M_1 and M_2 be two smooth open 4-manifolds. Let $\gamma_1:[0, \infty) \to M_1$, $\gamma_2:[0, \infty) \to M_2$ be smooth properly embedded rays with tubular neighborhoods ν_1 and ν_2, respectively. Let $M_1 \natural M_2 = M_1 \cup_{\phi_1} I \times \mathbf{R}^3 \cup_{\phi_2} M_2$, where ϕ_1: $[0, \frac{1}{2}) \times \mathbf{R}^3 \to \nu_1$ and ϕ_2: $(\frac{1}{2}, 1] \times \mathbf{R}^3 \to \nu_2$ are orientation-preserving diffeomorphisms which respect the $\mathbf{R}^3$-bundle structures. $M_1 \natural M_2$ is called the end-sum of M_1 and M_2 along γ_1 and γ_2 (see [4]). Note that $M_1 \natural M_2$ depends on γ_1 and γ_2.

Definition. Let M be a smooth, oriented, open 4-manifold with one end. M will be called end-periodic if there exists (1) a smooth unoriented manifold Y homeomorphic (but not necessarily diffeomorphic) to $S^3 \times S^1 \# n\mathbf{CP}^2$ for some finite n, where $n\mathbf{CP}^2$ denotes the connected sum of n copies of $\mathbf{CP}^2$, and (2) a neighborhood of the end of M which is diffeomorphic to a neighborhood of one end of $\tilde{Y}$, where $\tilde{Y}$ denotes the universal cover of Y (see [4]).

2. PROOF OF THEOREM 1

Let M be the total space of a smooth oriented $\mathbf{R}^2$-bundle over a closed connected oriented surface S. M has its usual smooth structure and orientation which are induced from the bundle structure.

To prove Theorem 1, we first state Taubes' theorem (cf. [4, 6]) in a form convenient for our purposes.

THEOREM (Taubes). *Let M be a smooth, simply connected, end-periodic 4-manifold. Suppose the intersection pairing on $H_2(M)$ is definite. Then this pairing is standard, i.e., diagonalizable over $\mathbf{Z}$.*

Let E_8 be the even, negative definite form of rank 8, and let Q_n be the standard, negative definite form of rank n (possibly infinite). Since the pairing $E_8 \oplus Q_n$ is not diagonalizable, $E_8 \oplus Q_n$ is not realized by an end-periodic 4-manifold M as above.

Proof of Theorem 1. Our method is essentially due to Gompf [4], who used it to produce an uncountable, doubly indexed family $\{R_{s,t} \mid 0 < s, t < \infty\}$ of exotic $\mathbf{R}^4$'s.

For a smooth manifold R which is homeomorphic to $\mathbf{R}^4$, let $f: R \to \mathbf{R}^4$ be a homeomorphism. Let $\gamma_1:[0, \infty) \to M$ and $\gamma_2:[0, \infty) \to R$ be smooth properly embedded rays. Assume γ_1 is a ray in one fiber. By Quinn [7], we may assume that f is smooth near γ_2. Using γ_1 and γ_2, we construct the end-sum $M \natural R$. Since the composite $f \circ \gamma_2$ and the positive x_1-axis of $\mathbf{R}^4$ are smoothly ambiently isotopic (by Gompf [4]), it follows that $M \natural R$ is homeomorphic to M.

In [1], Gompf constructs M_0 and R_Γ. The open 4-manifold M_0 is smooth and simply connected, with end collared topologically by $S^3 \times \mathbf{R}$. The intersection pairing of M_0 is $E_8 \oplus Q_1$. R_Γ, which is an exotic $\mathbf{R}^4$, is constructed so that a neighborhood U of its end is orientation- and end-preserving diffeomorphic to a neighborhood of the end of M_0.

Let $h: \mathbf{R}^4 \to R_\Gamma$ be a homeomorphism. By Quinn [7], we may assume that h is smooth near the positive x_1-axis. Let B_r denote the open ball of radius r about 0 in $\mathbf{R}^4$. Choose r_0 large enough that $h(B_{r_0}) \cup U = R_\Gamma$. Let $R_s = h(B_{r_0+s})$. Construct the end-sum $M \natural R_s$ along γ_1 and the image under h of the positive x_1-axis of $\mathbf{R}^4$. By the above argument, $M \natural R_s$ is homeomorphic to M.

M is the total space of a smooth oriented $\mathbf{R}^2$-bundle over a closed connected oriented surface S. Let χ denote the Euler class of the bundle. We first assume that $n = -\chi([S]) \geqslant 0$. Let $n\overline{\mathbf{CP}}^2$ denote $P_1 \# \cdots \# P_n$, where $P_1, \dots, P_n$ are n copies of $\overline{\mathbf{CP}}^2$. Let $g_1, \dots, g_n$ denote the images of the standard generators of $H_2(P_1; \mathbf{Z}), \dots, H_2(P_n; \mathbf{Z})$ in $H_2(n\overline{\mathbf{CP}}^2; \mathbf{Z})$, respectively. Then $g_1 + \cdots + g_n$ can be represented by a smoothly embedded 2-sphere, so it can be represented by a smoothly embedded surface which is diffeomorphic to S. Then there is an embedding of S in $n\overline{\mathbf{CP}}^2$ such that the Euler class of the normal bundle is χ. So there is an orientation-preserving diffeomorphism between M and a tubular neighborhood of S in $n\overline{\mathbf{CP}}^2$. Suppose $s < s'$. We will show that $M \natural R_{s'}$ does not embed in $M \natural R_s$.

Suppose $M \natural R_{s'}$ embeds (preserving orientation) in $M \natural R_s$. Because h is smooth near the positive x_1-axis, R_s embeds in $R_{s'}$ such that some neighborhood $U' \subset \partial R_s$ (the boundary of R_s in $R_{s'}$) is a smooth submanifold of $R_{s'}$. Because M is the total space of an $\mathbf{R}^2$-bundle, M', which is a copy of M, embeds in M such that $\partial M'$ is a smooth submanifold of M. Form the connected sum $M \# R_{s'}$ away from $R_s \cup \partial R_s$ and $M' \cup \partial M'$. Now find a smooth arc γ in $M \# R_{s'}$ which runs from the smooth part of ∂R_s to $\partial M'$ and hits $R_s \cup \partial R_s \cup M' \cup \partial M'$ only at the endpoints. The set $M' \cup R_s$ union a tubular neighborhood of γ is diffeomorphic to $M \natural R_s$. So $M \natural R_s$ embeds in $M \# R_{s'}$ with compact closure (cf. [4]).

We now have $R_{s'} \subset M \natural R_{s'} \hookrightarrow M \natural R_s \hookrightarrow M \# R_{s'} \hookrightarrow n\overline{\mathbf{CP}}^2 \# R_{s'}$, so there is an embedding i: $R_{s'} \hookrightarrow n\overline{\mathbf{CP}}^2 \# R_{s'}$ such that $i(R_{s'})$ has compact closure.

There is a neighborhood V of the end of $n\overline{\mathbf{CP}}^2 \# R_{s'}$ which is disjoint from $i(R_{s'})$ and $n\overline{\mathbf{CP}}^2$. We may assume that V is homeomorphic to $S^3 \times \mathbf{R}$. Let W denote $n\overline{\mathbf{CP}}^2 \# R_{s'}$ minus $i(R_{s'} - V)$, i.e., the region between V and $i(V)$. Then V and $i(V)$ are neighborhoods of the two ends of W, and we may identify these neighborhoods via i to obtain a closed, smooth manifold Y. By the annulus conjecture [7], Y is homeomorphic to $S^3 \times S^1 \# n\overline{\mathbf{CP}}^2$.

We may assume V lies in U, so there is an embedding $j: V \hookrightarrow M_0$, sending V "near" the end of M_0. Let M_1 denote M_0 minus the noncompact component of $M_0 - j(V)$. Form the manifold M_2 by gluing half of the universal cover of Y onto the end of M_1. That is, form M_2 from $M_1 \cup (W \times \{0, 1, 2, \dots\})$ by identifying $j(V)$ with $i(V) \times \{0\}$, and $V \times \{n\}$ with $i(V) \times \{n+1\}$ for $n = 0, 1, 2, \dots$. M_2 is simply connected and has intersection form $E_8 \oplus Q_\alpha$. But M_2 is clearly end-periodic, contradicting Taubes' theorem.

Now suppose $\chi([S]) > 0$. Then $\overline{M}$ is the total space of an $\mathbf{R}^2$-bundle over the oriented surface S. Let χ' denote the Euler class of the bundle. Then $\chi'([S]) < 0$. We can apply the previous argument to $\overline{M}$. Suppose $s < s'$. Then $\overline{M} \natural R_{s'}$ does not embed in $\overline{M} \natural R_s$ and $M \natural \overline{R}_{s'}$ does not embed in $M \natural \overline{R}_s$. This ends the proof of Theorem 1. □

Remark 1. Assume that there is a flat topological embedding of a closed connected orientable 3-manifold N in $n\overline{\mathbf{CP}}^2$ (ie., a topological embedding $i: N \times \mathbf{R} \hookrightarrow n\overline{\mathbf{CP}}^2$). By Quinn [7], one can assume that the embedding i is smooth near $p \times [-1, 0]$. (The smooth structure on $N \times \mathbf{R}$ is the standard product smooth structure and $p \in N$.) Let M' denote $i(N \times (-\infty, 0))$ (with the smooth structure induced from the smooth structure on $n\overline{\mathbf{CP}}^2$). Construct the end-sum $M' \natural R_s$ along $i(p \times [-1, 0))$ and the image under h of the positive x_1-axis (see the proof of Theorem 1). $M' \natural R_s$ is homeomorphic to $N \times \mathbf{R}$. If $s < s'$, then $M' \natural R_s$ embeds in $n\overline{\mathbf{CP}}^2 \# R_{s'}$ with compact closure. Applying the argument used in the proof of Theorem 1, we conclude that there are uncountably many nondiffeomorphic smooth structures on $N \times \mathbf{R}$. In particular, if N is a homology 3-sphere, then there are uncountably many nondiffeomorphic smooth structures on $N \times \mathbf{R}$.

Remark 2. Let N be a closed connected orientable 3-manifold. Assume that there is a topological embedding $N \times \mathbf{R} \hookrightarrow n\overline{\mathbf{CP}}^2$. Then one can obtain uncountably many smooth structures in each of the two concordance classes of smoothings of $N \times \mathbf{R}$ (which are distinguished by the Kirby–Siebenmann uniqueness obstruction in $H^3(N; \mathbb{Z}_2) \cong \mathbb{Z}_2$) (cf. [8]). This is because the smooth structure on $N \times \mathbf{R}$ induced from the given embedding $N \times \mathbf{R} \hookrightarrow n\overline{\mathbf{CP}}^2$ realizes one class, and the other is realized by the embedding $N \times \mathbf{R} \hookrightarrow (n+2)\overline{\mathbf{CP}}^2$ obtained by summing $n\overline{\mathbf{CP}}^2$ with one copy of the fake $\overline{\mathbf{CP}}^2$ (its Kirby–Siebenmann obstruction $ks \neq 0$) on each side of N, and identifying the result with the smooth manifold $(n+2)\overline{\mathbf{CP}}^2$ by Freedman's classification.

Remark 3. In fact, the method used in the proof of Theorem 1 can be applied to get the following:

Let M be a 4-dimensional smooth submanifold of $n\overline{\mathbf{CP}}^2$, $\partial M \neq \emptyset$. Let N be a closed subset of ∂M which contains a nonempty open subset of ∂M. Then $M - N$ admits uncountably many smooth structures.

More general results will be discussed elsewhere.

Remark 4. Let Σ denote the Poincaré homology 3-sphere, with its usual orientation as the boundary of an E_8 plumbing P. Then for each positive integer n, Σ cannot be smoothly embedded in $n\overline{\mathbf{CP}}^2$. If Σ can be smoothly embedded in $n\overline{\mathbf{CP}}^2$ for some n, it splits $n\overline{\mathbf{CP}}^2$ into two parts U and V. The intersection form of $n\overline{\mathbf{CP}}^2$, which is standard negative definite, decomposes as a direct sum of the intersection forms of U and V. So both U and V have standard negative definite intersection forms (rank zero is not excluded). (If a unimodular form is a summand of a standard negative definite form, this form is standard negative definite.) One of the boundaries of U and V (with the induced orientations from U and V, respectively), say U, is orientation-preserving diffeomorphic to $\bar{\Sigma}$. Glue U and P together along their boundaries. Let M denote the resulting smooth manifold. The intersection form of M is negative definite and has an E_8-summand, so it is not diagonalizable. This contradicts Donaldson's theorem on the nonexistence of certain closed, smooth 4-manifolds [9].

Remark 5. In [10], Freedman and Taylor construct a universal $\mathbf{R}^4$ U with the following property:

Let $e: U \to W$ be a smooth embedding of U into a smooth 4-manifold. There does not exist any topological embedding $h: D^4 \to W$ of the 4-cell into W with closure $(e(U)) \subset h(\text{interior } D^4)$.

Freedman and Taylor have conjectured that there does not exist a smooth embedding of U in any compact smooth 4-manifold. If this conjecture is true, then for any closed connected orientable 3-manifold N, $N \times \mathbf{R}$ (with its standard product smooth structure) is not diffeomorphic to $N \times \mathbf{R} \natural U$, since for some positive integer k, N can be smoothly embedded in $\#_k S^2 \times S^2$ (cf. [11]).

3. PROOF OF THEOREM 2

Proof of Theorem 2. Let N be a component of ∂M. Since every closed connected orientable 3-manifold bounds a compact simply connected 4-manifold, we can glue a compact simply connected 4-manifold onto each boundary component other than N to obtain a compact simply connected topological 4-manifold M_1 with boundary N. Set $M' = M_1 \cup_{id_{\partial M_1}} \overline{M}_1$. The Kirby–Siebenmann obstruction $ks(M') = 2ks(M_1) = 0$ (see [8]).

Let B be a smooth complex curve (Riemann surface) of degree $2p$ in the plane $\mathbf{CP}^2$. Let R_p denote the double covering of $\mathbf{CP}^2$ branched along B. R_p is simply connected and when p is even, the intersection form of R_p is odd. $b^+(R_p) = p^2 - 3p + 3$, $b^-(R_p) = 3p^2 - 3p + 1$ (see [12], Section 1.1.7).

Choose p large enough that $b^+(R_p) > b^+(M')$ and $b^-(R_p) > b^-(M')$. When p is even, $M' \# [b^+(R_p) - b^+(M')]\mathbf{CP}^2 \# [b^-(R_p) - b^-(M')]\overline{\mathbf{CP}}^2$ is homeomorphic to R_p by Freedman's work [2]. Hence, the interior of M admits a topological embedding into R_p and has a complex structure induced from the complex structure on R_p. This finishes the proof of Theorem 3. $\square$

Example. There does not exist any complex structure on the oriented topological manifold $\mathbf{CP}^2 \# \mathbf{CP}^2$. But we can apply the previous argument to obtain a complex structure on the oriented topological manifold $\mathbf{CP}^2 \# \mathbf{CP}^2 -$ pt. Let C denote $\mathbf{CP}^2 \# \mathbf{CP}^2$-pt. with the smooth structure induced from the complex structure. There does not exist any orientation-preserving diffeomorphism between C and $\mathbf{CP}^2 \# \mathbf{CP}^2$-pt. (with its usual smooth structure), however, since, by Donaldson's work [13], there does not exist any orientation-preserving diffeomorphism between R_p and $M_1 \# M_2$, where $M_1 = \mathbf{CP}^2 \# \mathbf{CP}^2$ and $b^+(M_2) > 0$.

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Department of Mathematics
Zhejiang University
Hangzhou 310027
Peoples' Republic of China